Nash equilibrium of Google Auction

- **Hal Varian**
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- **Abstract**

I analyze the Google Ad words auction using game theory. Given a set of advertisers and their click-values, I can solve for an equilibrium set of bids. I can also invert this calculation: given a set of bids, I can find the set of click-values that are consistent with those bids, if such a set exists. The Nash equilibrium model fits the data very well, and yields plausible click-valuations, suggesting that the model does a good job of explaining the data.

Simplified auction rules

1) Advertiser i has value for click v_i, for $i = 1$, ..., n.

2) Advertiser i announces bid b_i, $i = 1$, ..., n.

3) Highest bidder gets position 1, second highest position 2, and so on.

4) Bidder in position i pays price-per-click   determined by bid of advertiser below him, so $p_i = b_{i+1}$.

5) Payoff to bidder i is then $(v_i - b_{i+1})\, x_i$, where x_i is the CTR for position i.

(The real auction orders advertiser by ad quality × bids. This adds a few constants as described below, but the basic strategic analysis is simpler if we set ad quality = 1.)

Example

psn	value	ctr	bid	profit
1	v_1	x_1	b1	$(-b2 + v_1)\ x_1$
2	v_2	x_2	b2	$(-b3 + v_2)\ x_2$
3	v_3	x_3	b3	$(-b4 + v_3)\ x_3$
4	v_4	0	b4	0

Note that if bidder 3 wanted to move up to position 2, he would have to bid $b_2 + e$, while if 2 wanted to move to position 3 he would have to bid $b_4 + e$.

To move up you have to beat the *bid* a bidder is making, to move down you have to beat the *price* a bidder is paying.

Nash equilibrium

A set of bids (b_i) is a *Nash equilibrium* if bidder i makes at least as much profit by being in position i than in any other position j, assuming the other bidders don't change their behavior. That is:

$$(v_i - b_{i+1})\, x_i \geq \left(v_i - b_j\right) x_j \quad \text{for } j < i \qquad \text{(NE1)}$$

$$(v_i - b_{i+1})\, x_i \geq \left(v_i - b_{j+1}\right) x_j \quad \text{for } j > i \qquad \text{(NE2)}$$

Manipulate Nash inequalities

Can rearrange to:

$$v_i(x_i - x_j) + b_j x_j \geq b_{i+1} x_i \text{ for } j < i \quad \text{(NE2a)}$$
$$v_i(x_i - x_j) + b_{j+1} x_j \geq b_{i+1} x_i \text{ for } j > i. \quad \text{(NE2b)}$$

Any set of (b_i) that satisfy these linear inequalities is an equilibrium. Ususally, there will be an entire range of such solutions. Can use linear programming to solve for the Nash equilibrium set of bids that yield the maximum or minimum revenue to search engine.

$\max_b \sum b_{i+1} x_i$ such that NE1a and NE2a are satisfied
$\min_b \sum b_{i+1} x_i$ such that NE1a and NE2a are satisfied

But there is an easier way to find an interesting subset of the NE.

Symmetric Nash equilibrium

Suppose we find a set of (b_i) that satisfies the *symmetric Nash inequalities*

$$b_1 > b_2 > ... > b_n \qquad \text{(SNE1)}$$

$$v_i(x_i - x_j) + b_{j+1} x_j \geq b_{i+1} x_i \text{ for all } i \text{ and } j. \qquad \text{(SNE2)}$$

(These are just the $j > i$ inequalities we saw before, but now written for all i and j.)

Then we must also satisfy (NE1a-b) since:

$$v_i(x_i - x_j) + b_j x_j \geq v_i(x_i - x_j) + b_{j+1} x_j \geq b_{i+1} x_i \text{ for } j < i.$$

Hence, the set of $b_i s$ that solve (SNE1-2) are a subset of the set of $b_i s$ that satisfy (NE1-2). It turns out to be a very nicely behaved subset closely related to the classical assignment problem.

One-step inequalities

Writing these inequalities for $j = i + 1$ and $j = i - 1$ we have:

$v_{i-1}(x_{i-1} - x_i) + b_{i+1} x_i \geq b_i x_{i-1} \geq v_i(x_{i-1} - x_i) + b_{i+1} x_i$

We can show that if you satisfy the one-step inequalities you satisfy them all. The above inequalities give us a nice recursion to solve for a Nash equilibrium. The smallest and largest (b_i) that solve (SNE1-2) are therefore the solutions to these recursions:

$$b_i x_{i-1} = b_{i+1} x_i + v_i(x_{i-1} - x_i) \qquad \text{(RNE1)}$$
$$b_i x_{i-1} = b_{i+1} x_i + v_{i-1}(x_{i-1} - x_i). \qquad \text{(RNE2)}$$

RNE1 is the same as the no-envy equilibrium of Edelman, Ostrovsky, and Schwartz: "Internet Advertising and the Generalized Second Price Auction", NBER Working Paper, 2005.

Bidding functions

Take the lower bound:

$$b_i\, x_{i-1} = b_{i+1}\, x_i + v_i(x_{i-1} - x_i) \qquad \text{(RNE1)}$$

Divide through by x_{i-1}. Letting $a_i = x_i / x_{i-1}$ we have:

$$b_i = b_{i+1}\, a_i + v_i(1 - a_i) \qquad \text{(RNE1)}$$

So the equilibrium bid of agent *i* is a convex combination of his value and the bid of the agent below him. First *excluded* bidder bids his true value. Can solve the recursion by repeated substitution to get explicit solution:

$$b_{i+1}\, x_i = \sum_{j>i} v_j\left(x_{j-1} - x_j\right)$$

Total revenue $= v_2\,(x_1 - x_2) + 2\, v_3\,(x_2 - x_3) + 3\, v_4\,(x_3 - x_4)$ ...

From bids to values

We have seen that a symmetric Nash equilibrium satisfies these inequalities:

$$b_i\, x_{i-1} \geq v_i (x_{i-1} - x_i) + b_{i+1}\, x_i$$

$$b_{i+1}\, x_i \leq v_i (x_i - x_{i+1}) + b_{i+2}\, x_{i+1}$$

Rearranging gives:

$$\frac{b_i\, x_{i-1} - b_{i+1}\, x_i}{x_{i-1} - x_i} \geq v_i \geq \frac{b_{i+1}\, x_i - b_{i+2}\, x_{i+1}}{x_i - x_{i+1}}$$

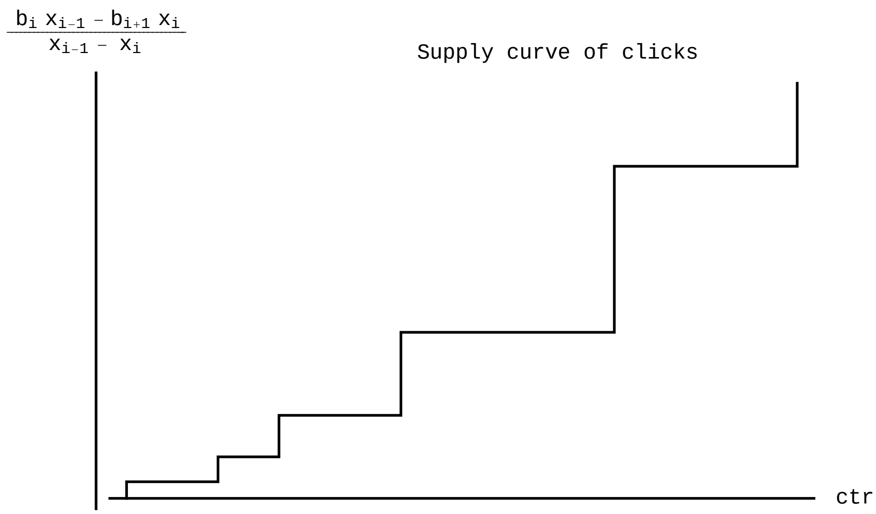
Bounds can be interpreted as:

$$\frac{b_i\, x_{i-1} - b_{i+1}\, x_i}{x_{i-1} - x_i} = \frac{\text{change in costs}}{\text{change in clicks}} = \text{incremental cost per click}$$

This gives intuitive interpretation of bounds: move up till the incremental cost per click exceeds your value per click.

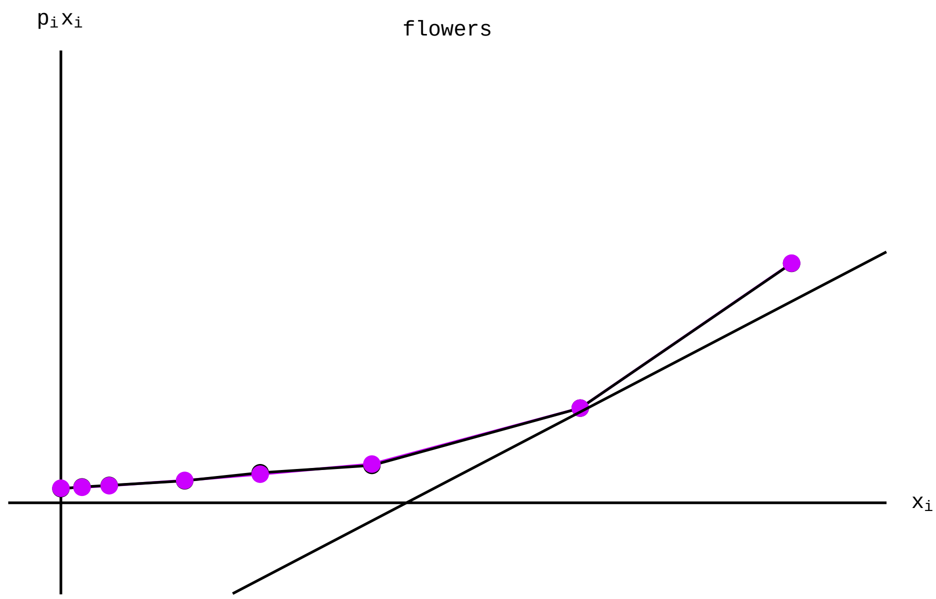
Interpretation of SNE conditions

SNE inequalities say that in equilibrium the *incremental cost per click must increase with the click-through-rate.* Why? Consider the first time it decreased. At this point bidder bought expensive clicks and then refused to buy cheaper incremental clicks. That cannot be an equilibrium.



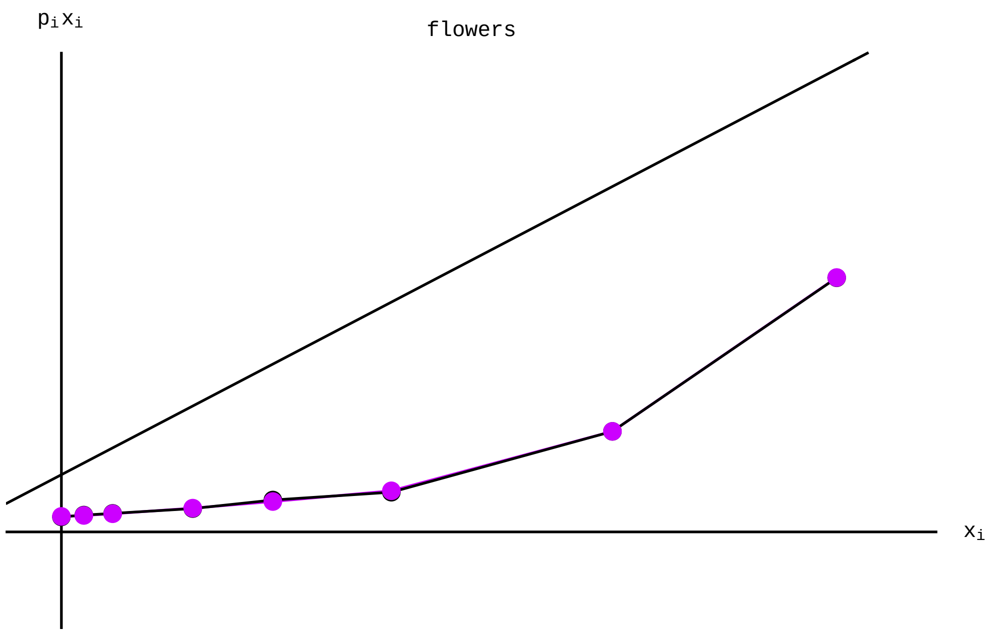
Geometry of conditions

Plot the expenditure profile $b_{i+1}\, x_i = p_i\, x_i$ versus x_i. The slopes of the line segments emanating from each point are the incremental costs per click. The slope of the supporting lines at each vertex are the possible click-values associated with that bidder.



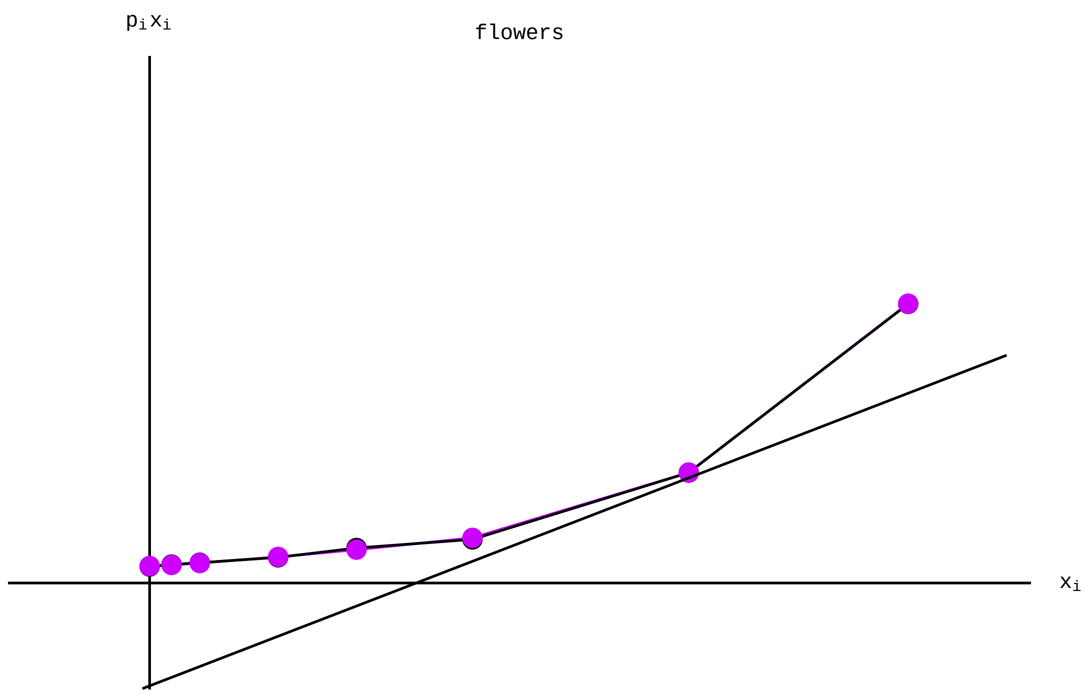
Maximizing profit

Profit for advertiser i is: $\pi_i = v_i\, x_i - p_i\, x_i$. This is a straight line in $(x_i,\ p_i\, x_i)$ space with vertical intercept equal to the negative of profit. So maximizing profit means shifting this isoprofit line as far down as possible:



Range of values = range of slopes

Furthermore, the range of click-values for which a given position is optimal is given by the slope of the supporting line:



Quality adjustment

We view the actual CTR, z_i, to be the product of the position-specific effect, x_i, and a quality effect e_i, so that $z_i = e_i x_i$. If an advertiser changes position, the advertiser-specific effect (e_i) goes with him. We order the bidders by $e_i b_i$ and the "bid discounter" sets the price that i pays to be just enough to retain his position. That is, $p_i e_i = b_{i+1} e_{i+1}$, or $p_i = b_{i+1} e_{i+1} / e_i$. Nash equilibrium then entails the inequalities:

$$(v_i - b_{i+1} e_{i+1} / e_i) e_i x_i \geq (v_i - b_j e_j / e_i) e_i x_j \quad \text{for } j < i$$
$$(v_i - b_{i+1} e_{i+1} / e_i) e_i x_i \geq (v_i - b_{j+1} e_{j+1} / e_i) e_i x_j \quad \text{for } j > i$$

As before, we also have a symmetric version of the inequalities

$$(e_i v_i - b_{i+1} e_{i+1}) x_i \geq (e_i v_i - b_{j+1} e_{j+1}) x_j \quad \text{for all } i \text{ and } j.$$

Empirical implications

If the data come from a Nash equilibrium, then we must satisfy

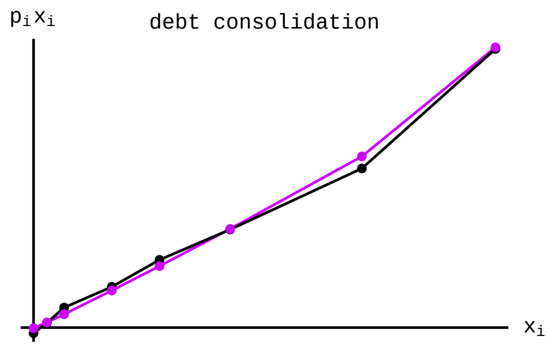
$$\frac{b_i\, e_i\, x_{i-1} - b_{i+1}\, e_{i+1}\, x_i}{x_{i-1} - x_i} \geq e_i\, v_i \geq \frac{b_{i+1}\, e_{i+1}\, x_i - b_{i+2}\, e_{i+2}\, x_{i+1}}{x_i - x_{i+1}}$$

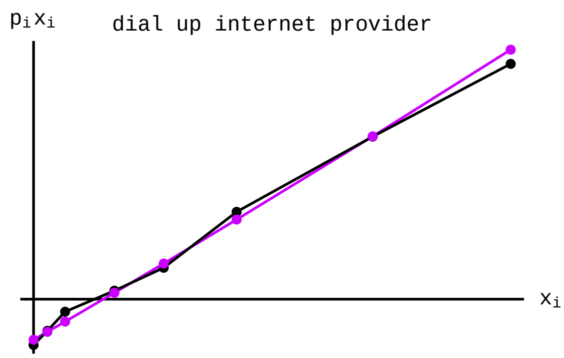
Furthermore, if these intervals are non-empty, we can always find a set of (v_i) that are consistent with the observed bids.

So a necessary and "sufficient" test for Nash equilibrium is to see whether the incremental cost per click is increasing in the CTR.

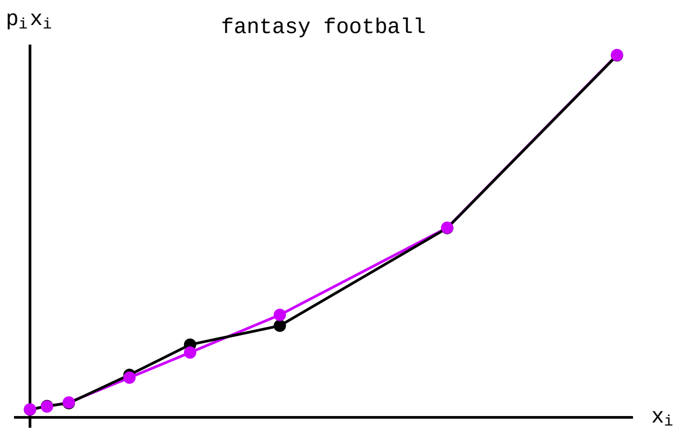
Perturbing the data

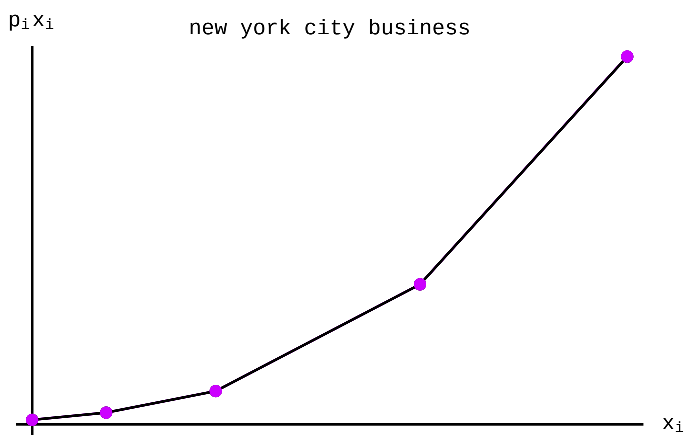
If the data don't already satisfy the SNE inequalities, we can ask for the minimal perturbation (in terms of sum-of-squared residuals) that does satisfy the inequalities. This is a quadratic programming problem.





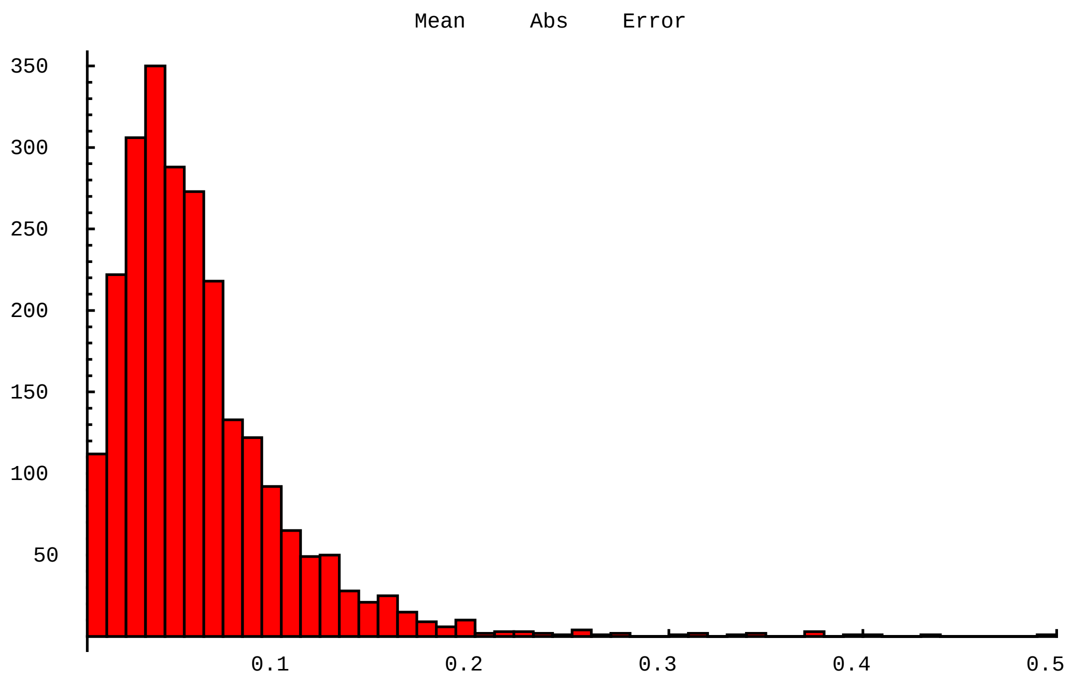
Other cases





Quantifying the fit

I looked at 2425 randomly chosen keyword auctions and found that the average absolute deviation required to satisfy the inequalities was about 5.8% (median 4.8%). It was almost always less than 10%.



Bounds on values

Can use the fitted convex function to estimate the click-values:

	rawLB	rawUB	qpLB	qpUB	price
1	0.64	∞	0.76	∞	0.6
2	0.57	0.54	0.63	0.65	0.5
3	0.53	0.38	0.42	0.42	0.32
4	0.25	0.42	0.32	0.33	0.22
5	0.24	0.26	0.33	0.33	0.22
6	1.06	0.42	0.57	0.57	0.38
7	0.67	0.7	0.37	0.38	0.22
8	0.05	1.25	0.05	0.69	0.36

Bounds on values, cont.

	rawLB	rawUB	qpLB	qpUB	price
1	0.6	∞	0.78	∞	0.56
2	0.53	0.47	0.59	0.6	0.43
3	1.39	0.91	0.99	1.01	0.68
4	0.92	1.42	0.94	1.01	0.55
5	0.77	1.24	1.14	1.26	0.66
6	0.5	0.5	0.73	0.74	0.41
7	1.39	0.37	0.53	0.54	0.3
8	0.05	1.35	0.05	0.51	0.23

Bounds on values, cont.

1) Nash equilibrium describes the data well.

2) Values are usually reasonable, often roughly twice the bids.

3) Position RHS1 often has a high value, possibly due to promotion policy (which isn't modeled here), or to preference for first position, or for impression value.

4) If the expenditure profile is flat, the clicks values are about the same. If it is highly convex, the values are quite different.

Incentives

The Google auction does not result in truthful bidding. What about VCG auction which has truthful revelation?

Each advertiser reports his value-per-click, r_i. Google assigns advertisers to positions to maximize value of page. Payment of advertiser i = value accruing to other advertisers if i is present − value accruing to other advertisers if i is absent.

If i is absent, each advertiser below him shifts up by 1 slot, so i's payment is:

$$\text{payment by } i = \sum_{j>i} r_j (x_{j-1} - x_j)$$

Note that when $r_i = v_i$ this is the same as the lower bound on the SNE! This is true more generally; see Demange and Gale (1985) and Roth and Sotomayor (1990).